Quadratic forms and Galois cohomology

The connection (Milnor's conjectures)

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Description via symbols

Let F be a field with $\mathrm{char}(F) \neq 2$.

$$\overbrace{(F^\times)/(F^\times)^2 \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} (F^\times)/(F^\times)^2}^{n \text{ times}}$$

$$\swarrow \qquad\qquad \downarrow \qquad\qquad \searrow \ \text{induced by the cup product}$$

$$I^n(F)/I^{n+1}(F) \xleftarrow{\ s_n\ } K_n^M(F)/2 \xrightarrow{\ h^n_{F,2}\ } H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n})$$

with

$$s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F), \quad \{a_1, \ldots, a_n\} \mapsto (\langle a_1 \rangle - \langle 1 \rangle) \cdot \ldots \cdot (\langle a_n \rangle - \langle 1 \rangle)$$

Question

Are s_n and $h^n_{F,2}$ isomorphisms?

Proof that s_n is well-defined and surjective

Proposition

The map $s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ is well-defined and surjective.

Proof.

Consider the map $F^\times \times \ldots \times F^\times \to I^n(F)/I^{n+1}(F), \quad (a_1, \ldots, a_n) \mapsto (\langle a_1 \rangle - \langle 1 \rangle) \cdot \ldots \cdot (\langle a_n \rangle - \langle 1 \rangle)$.

n-linear: $\langle a \rangle - \langle 1 \rangle + \langle b \rangle - \langle 1 \rangle = \langle ab \rangle - \langle 1 \rangle \mod I^2(F)$, because
$\langle a \rangle + \langle b \rangle - \langle ab \rangle - \langle 1 \rangle = -(\langle a \rangle - \langle 1 \rangle)(\langle b \rangle - \langle 1 \rangle) \in I^2(F)$.

Steinberg: $(\langle a \rangle - \langle 1 \rangle)(\langle 1-a \rangle - \langle 1 \rangle) = \langle a(1-a) \rangle - \langle a \rangle - \langle 1-a \rangle + \langle 1 \rangle = 0$, because
$\langle a \rangle + \langle 1-a \rangle = \langle a + (1-a) \rangle + \langle a(1-a)(a+1-a) \rangle = \langle 1 \rangle + \langle a(1-a) \rangle$.

mod 2: $2\{a_1, \ldots, a_n\} = \{a_1^2, a_2, \ldots, a_n\} \mapsto \underbrace{(\langle a_1^2 \rangle - \langle 1 \rangle)}_{=\langle 1 \rangle - \langle 1 \rangle = 0}(\langle a_2 \rangle - \langle 1 \rangle) \cdot \ldots \cdot (\langle a_n \rangle - \langle 1 \rangle) = 0$.

surjective: $I(F)$ is additively generated by Pfister forms $\langle a \rangle - \langle 1 \rangle$.

□

Splitting the problem into two

Let F be a field with $\mathrm{char}(F) \neq 2$.

Conjecture ('Milnor conjecture on norm residue symbol', 'Bloch-Kato conjecture for prime 2')

The map $h^*_{F,2} \colon K^M_*(F)/2 \longrightarrow H^*_{\mathrm{Gal}}(F, \mu_2^{\otimes *})$ is an isomorphism.

Conjecture ('Milnor conjecture on quadratic forms')

The map $s_* \colon K^M_*(F)/2 \longrightarrow \mathrm{gr}_I(W(F)) = \bigoplus_{n \in \mathbb{N}_0} I^n(F)/I^{n+1}(F)$ is an isomorphism.

Proof strategy for the Milnor conjecture on norm residue symbol

Want to show: The map $h^n_{F,2}\colon K^M_n(F)/2 \to H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n})$ is an isomorphism.

1) Induction on n: If the statement hold for all fields F and $n < N$ then it holds for $n = N$.
2) $h^n_{F,2}$ is an isomorphism for certain 'big' fields F (i.e. F has no odd degree extensions and $K^M_n(F) = 2K^M_n(F)$)
3) Assume there is a field F for which $h^n_{F,2}$ is not an isomorphism, then there is an extension providing a counter example to the previous step.

Details: For any $\{a_1, \dots, a_n\} \in K^M_n(F)$ there is a field extension $F \hookrightarrow F'$ such that $\{a_1, \dots, a_n\} \in 2K^M_n(F')$ and $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$ is not an isomorphism (take a big colimit to get a single field providing a counter example).

On the third step: how to find a good field extension

Suppose there is a field F for which $h^n_{F,2}$ is not an isomorphism.

> Goal: for a symbol $\{a_1, \dots, a_n\} \in K^M_n(F)$ find a field extension F' such that $\{a_1, \dots, a_n\} \in 2K^M_n(F')$ and $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$ is not an isomorphism.

The first part is easy: take $F' = F[X]/(X^2 - a_i)$ for any $i = 1, \dots, n$.
The problem is to control $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$.
Instead, use $F(Q_{\{a_1,\dots,a_n\}})$ with

$$Q_{\{a_1,\dots,a_n\}} = \{q_{\langle\langle a_1\rangle\rangle\otimes\dots\otimes\langle\langle a_{n-1}\rangle\rangle}(x_0, \dots, x_{2^{n-1}-1}) - a_n x^2_{2^{n-1}} = 0\} \subseteq \mathbb{P}^{2^{n-1}}_F$$

i) Introduce motivic cohomology to study the behaviour of $K^M_n(F)/2 \to K^M_n(F')/2$ and $H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n}) \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$

ii) Algebraic topological input: motivic Steenrod operations

iii) Algebraic geometry input: identify a direct summand of the motive of $Q_{\{a_1,\dots,a_n\}}$

Starting the proof of the Milnor conjecture on quadratic forms

Want to show: The map $s_n\colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ is an isomorphism.

Known cases are:

i) The map s_n is surjective for all $n \in \mathbb{N}$ (see before).

ii) The maps $s_0\colon \mathbb{Z}/2\mathbb{Z} \to W(F)/I(F)$ and $s_1\colon F^\times/(F^\times)^2 \to I(F)/I^2(F)$ are isomorphisms

iii) The map s_2 is an isomorphism (by writing down an inverse)

Using 'standard' facts about quadratic forms (Arason-Pfister Hauptsatz, ...) one can show:

Proposition

$$s_n(\{a_1, \dots, a_n\}) = s_n(\{b_1, \dots, b_n\}) \quad \Leftrightarrow \quad \{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \in K_n^M(F)/2$$

This does not show the injectivity of s_n. It shows injectivity for pure symbols.

Proof strategy of the Milnor conjecture on quadratic forms

Have: Injectivity of $s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ on pure symbols

Idea: Find a field extension F' such that $\alpha \in K_n^M(F)/2$ becomes a pure symbol in $K_n^M(F')/2$.
Observation: By going to $F(Q_{\{a_1,\dots,a_n\}})$ the symbol $\{a_1, \dots, a_n\}$ vanishes.
Key: The kernel $K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2$ is a s nice as possible.

Proposition

If $\alpha = \{a_1, \dots, a_n\} \neq 0 \in K_n^M(F)/2$, then

$$\ker\left(K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2\right) = \mathbb{Z}/2\mathbb{Z} \cdot \alpha.$$

For $\alpha = \alpha_1 + \dots + \alpha_k \in K_n^M(F)/2$ take $F' = F(Q_{\alpha_1})(Q_{\alpha_2}) \dots (Q_{\alpha_i})$ such that

$$\alpha \neq 0 \in K_n^M(F')/2 \text{ and } \alpha = 0 \in K_n^M(F'(Q_{\alpha_{i+1}}))/2.$$

Understanding the kernel

Proposition

If $\alpha = \{a_1, \dots, a_n\} \neq 0 \in K_n^M(F)/2$, then

$$\ker \left(K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2 \right) = \mathbb{Z}/2\mathbb{Z} \cdot \alpha.$$

The maps $K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2$ already played a crucial role in the previous proof.

i) Describe the kernel in terms of a motivic cohomology group.

ii) Again use the splitting of the motive of Q_α.

Summary

1) The connection between quadratic forms and Galois cohomology has the form

$$I^n(F)/I^{n+1}(F) \cong K_n^M(F)/2 \cong H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n}),$$

2) These objects have a nice description via symbols modulo a single relation in degree 2

$$K_n^M(F) = (F^\times)^{\otimes n}/\langle \ldots \otimes a \otimes 1 - a \otimes \ldots \rangle,$$

3) The proofs require heavy, but interesting, machinery (e.g. motivic cohomology).