Tensor Triangulated Categories

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Definition (Triangulated Category)

A triangulated category $\mathcal{K}$ is an additive (essentially small) category together with a 'shift' $\Sigma\colon \mathcal{K} \to \mathcal{K}$ and a collection of distinguished triangles $\Delta = \left(a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \right)$ such that:

Bookkeeping: Δ distinguished and $\Delta \simeq \Delta'$ implies Δ' distinguished,

$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ is distinguished if and only if $b \xrightarrow{g} c \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma f} \Sigma b$ is,

$a \xrightarrow{1} a \longrightarrow 0 \longrightarrow \Sigma a$ is distinguished

Existence: Every $a \xrightarrow{f} b$ extends to a distinguished triangle

Morphism: Every partial morphism between distinguished triangle extends as follows

$$\begin{array}{ccccccc} a & \xrightarrow{f} & b & \xrightarrow{g} & c & \xrightarrow{h} & \Sigma a \\ \downarrow k & & \downarrow l & & \downarrow \exists m & & \downarrow \Sigma k \\ a' & \xrightarrow[f']{} & b' & \xrightarrow[g']{} & c' & \xrightarrow[h']{} & \Sigma a' \end{array}$$

Octaheder: . . .

Definition

A functor $T\colon \mathcal{K} \to \mathcal{K}'$ between triangulated categories is called exact/triangular if it commutes with shifts (i.e. $T\Sigma \simeq \Sigma' T$) and preserves distinguished triangles.

Definition

A tensor triangulated category $\mathcal{K}$ is a triangulated category with a monoidal structure $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ (i.e. $\otimes$ is 'associative' and 'has a unit $1 = 1_{\mathcal{K}}$') such that:
$-\otimes-$ is exact in each variable (i.e. $-\otimes a\colon \mathcal{K} \to \mathcal{K}$ and $a \otimes -\colon \mathcal{K} \to \mathcal{K}$ are exact for every $a \in \mathcal{K}$).

Remark

Additionally we assume that the monoidal structure is symmetric, i.e. $a \otimes b \simeq b \otimes a$.

Remark

There a certain compatibility assumptions hiding. E.g.

$$\begin{array}{ccccc} & & \Sigma\left(a \otimes (\Sigma b)\right) & & \\ & \nearrow & & \searrow & \\ (\Sigma a) \otimes (\Sigma b) & & & & \Sigma^2\left(a \otimes b\right) \\ & \searrow & & \nearrow & \\ & & \Sigma\left((\Sigma a) \otimes b\right) & & \end{array}$$

both ways give elements in $\mathrm{Hom}_{\mathcal{K}}\left((\Sigma a) \otimes (\Sigma b),\ \Sigma^2(a \otimes b)\right)$*, which are assumed to only differ by a sign.*

Definition

An exact functor F between tensor triangulated categories is called $\otimes$-exact if it preserves the tensor structure (including the unit) up to isomorphism.

Remark

Again with certain compatibility conditions:

$$
\begin{array}{ccccccc}
 & & F\left(\Sigma\left(a \otimes b\right)\right) & \longrightarrow & \Sigma\left(F\left(a \otimes b\right)\right) & & \\
 & \nearrow & & & & \searrow & \\
F\left((\Sigma a) \otimes b\right) & & & & & & \Sigma\left((Fa) \otimes (Fb)\right) \\
 & \searrow & & & & \nearrow & \\
 & & \left(F\left(\Sigma a\right)\right) \otimes (Fb) & \longrightarrow & (\Sigma(Fa)) \otimes (Fb) & &
\end{array}
$$

Note that every morphism in the diagram above is an isomorphism (but not necessarily unique).

Definition

A non-empty full subcategory $\mathcal{J} \subseteq \mathcal{K}$ is called:

- triangulated: If for every distinguished triangle $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ in $\mathcal{K}$ with two of a, b, c in $\mathcal{J}$, all three have to lie in $\mathcal{J}$.
- thick: If it is triangulated and $a \oplus b \in \mathcal{J} \implies a, b \in \mathcal{J}$.
- $\otimes$-ideal: If $\mathcal{K} \otimes \mathcal{J} \subseteq \mathcal{J}$.
- radical: If $a^{\otimes n} \in \mathcal{J} \implies a \in \mathcal{J}$.

Remark

A triangulated subcategory $\mathcal{J} \subseteq \mathcal{K}$ is additive and replete, i.e. $a \simeq b \in \mathcal{J} \implies a \in \mathcal{J}$.
Quick proof:

- *additive: Full, non-empty, Bookkeeping (3) and triangulated $\implies$ pre-additive*
  sum of dist. triangles is dist. (Existence, Morphism, Bookkeeping (1) and "four-lemma")

  $a \longrightarrow a \longrightarrow 0 \longrightarrow \Sigma a$ *by Bookkeeping (3)*

  $0 \longrightarrow b \longrightarrow b \longrightarrow 0$ *by Bookkeeping (2 and 3)*

  their sum and triangulated $\implies$ additive
- *replete: Bookkeeping (1 and 3) and triangulated* $a \xrightarrow{\sim} b \longrightarrow 0 \longrightarrow \Sigma a$

Definition

Let X be a quasi-compact and quasi-separated scheme. Denote by $\mathrm{D}(\mathrm{QCoh}(X))$ the derived category of quasi-coherent sheaves on X.

Define $D^{\mathrm{perf}}(X)$ as the full subcategory of $\mathrm{D}^b(\mathrm{Coh}(X)) \subseteq \mathrm{D}(\mathrm{QCoh}(X))$ of perfect complexes (i.e. complexes locally quasi-isomorphic to a bounded complex of free sheaves of finite rank)

Example

Let X be a variety (separated, finite type over a field k)

- X quasi-projective: $\mathrm{D}^{\mathrm{perf}}(X) = \mathrm{D}^b(\mathrm{VB}_X)$.
- $X = \mathrm{Spec}(R)$ affine: $\mathrm{D}(\mathrm{QCoh}(X)) \cong \mathrm{D}(R\text{-Mod})$ and $D^{\mathrm{perf}}(X) \cong \mathrm{K}^b(R\text{-proj})$.

Definition

An object C is called compact, if the functor $\mathrm{Hom}(C, -)$ commutes with arbitrary coproducts.

Theorem

Let X be a quasi-compact, quasi-separated scheme. The compact objects in $\mathrm{D}(\mathrm{QCoh}(X))$ *are precisely the objects* $D^{\mathrm{perf}}(X)$

Proof

First reduce to the case that $X = \mathrm{Spec}(R)$ *is affine, i.e.* $\mathrm{D}(\mathrm{QCoh}(X)) = \mathrm{D}(R\text{-}Mod)$.
Idea: both are equal to $\mathrm{Thick}(R)$, *the smallest thick subcategory of* $\mathrm{D}(R\text{-}Mod)$ *containing* R.

- $\mathrm{Thick}(R) =$ *compacts: This is a theorem.*
- $\mathrm{Thick}(R) \subseteq$ *compacts:* $\mathrm{Hom}_{\mathrm{D}(R\text{-}Mod)}(R, L) \cong \mathrm{Hom}_{\mathrm{K}(R\text{-}Mod)}(R, L) \cong H^0(L)$, *so* R *is compact and the compact objects form a thick subcategory.*
- $\mathrm{Thick}(R) \subseteq \mathrm{D}^{perf}(X)$: R *is perfect and* $\mathrm{D}^{perf}(X)$ *is a thick subcategory.*
- $\mathrm{D}^{perf}(X) \subseteq \mathrm{Thick}(R)$: $\mathrm{Thick}(R)$ *is additive and contains* R, *thus* R^n *for all* n.
  It also contains all finitely generated projectives (thick) and their shifts (triangulated).
  Let $P := 0 \to P^i \to \cdots \to P^s \to 0$ *be a perfect complex. This gives a short exact sequence of complexes of* R*-modules.*

$$0 \to P^s[s] \to P \to P^{\leq s-1} \to 0$$

  This gives a distinguished triangle in $\mathrm{D}(R\text{-}Mod)$

$$P^s[s] \to P \to P^{\leq s-1} \to \Sigma(P^s[s]).$$

By induction (triangulated) this reduces to the case that $\mathrm{Thick}(R)$ *contains all all finitely generated projectives and their shifts (done). This shows that* $\mathrm{Thick}(R)$ *contains all perfect complexes (repleted).*

□

Definition

A thick $\otimes$-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is called prime if:

- it is proper (i.e. $1_{\mathcal{K}} \notin \mathcal{P}$),
- $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Definition

The spectrum of $\mathcal{K}$ is the set of primes:

$$\mathrm{Spc}(\mathcal{K}) := \{\mathcal{P} \subseteq \mathcal{K} | \ \mathcal{P} \text{ is prime}\}.$$

Definition

For any family of objects $\mathcal{S} \subseteq \mathcal{K}$ define:

$$\mathrm{Z}(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | \ \mathcal{S} \cap \mathcal{P} = \emptyset\}.$$

Remark (Important!)

This is not the definition you know from algebraic geometry!
This would look like:

$$\mathrm{V}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{K}) | \ \mathcal{S} \subseteq \mathcal{P}\}.$$

Proposition

Let $\mathcal{K}$ be a tensor triangulated category and $\mathcal{S}_j \subseteq \mathcal{K}$ families of objects for $j \in J$. Then:

- *1):* $\mathrm{Z}(\mathcal{K}) = \emptyset$ *and* $\mathrm{Z}(\emptyset) = \mathrm{Spc}(\mathcal{K})$
- *2):* $\mathcal{S}_i \subseteq \mathcal{S}_j \implies Z(\mathcal{S}_j) \subseteq Z(\mathcal{S}_i)$
- *3):* $\mathrm{Z}(\mathcal{S}_i) \cup \mathrm{Z}(\mathcal{S}_j) = \mathrm{Z}(\mathcal{S}_i \oplus \mathcal{S}_j) \left(" = \mathrm{V}(\mathcal{S}_i \cap \mathcal{S}_j)" \right)$ *for* $\mathcal{S}_i \oplus \mathcal{S}_j = \{a_i \oplus a_j |\ a_i \in \mathcal{S}_i, a_j \in \mathcal{S}_j\}$
- *4):* $\bigcap_{j \in J} \mathrm{Z}(\mathcal{S}_j) = \mathrm{Z}(\bigcup_{j \in J} \mathcal{S}_j)$

Proof

Recall: $\mathrm{Z}(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) |\ \mathcal{S} \cap \mathcal{P} = \emptyset\}$.

- *1),2),4): clear (although different argument for 1) compared to AG)*
- *3):* $a_i \oplus a_j \in \mathcal{P} \cap (\mathcal{S}_i \oplus \mathcal{S}_j)$, *then* $a_i, a_j \in \mathcal{P}$ *(by thickness) and hence* $\mathcal{P} \notin \mathrm{Z}(\mathcal{S}_i) \cup \mathrm{Z}(\mathcal{S}_j)$,
  If there are $a_i \in \mathcal{P} \cap \mathcal{S}_i$ *and* $a_j \in \mathcal{P} \cap \mathcal{S}_j$ *then* $a_i \oplus a_j \in \mathcal{P} \cap (\mathcal{S}_i \oplus \mathcal{S}_j)$ *(by additivity)*

Remark

This defines the Zariski topology on $\mathrm{Spc}(\mathcal{K})$*, where the closed sets are given by* $\mathrm{Z}(\mathcal{S})$ *for* $\mathcal{S} \subseteq \mathcal{K}$*.*
Denote the open complement of $\mathrm{Z}(\mathcal{S})$ *by* $\mathrm{U}(\mathcal{S})$*:*

$$\mathrm{U}(\mathcal{S}) := \mathrm{Spc}(\mathcal{K}) \setminus \mathrm{Z}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) |\ \mathcal{S} \cap \mathcal{P} \neq \emptyset\}$$

Proposition

The open sets $\mathrm{U}(a) := \mathrm{U}(\{a\}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) |\ a \in \mathcal{P}\} \subseteq \mathrm{Spc}(\mathcal{K})$ *satisfy the following:*

- *1):* $\mathrm{U}(0) = \mathrm{Spc}(\mathcal{K})$ *and* $\mathrm{U}(1) = \emptyset$
- *2):* $\mathrm{U}(a \oplus b) = \mathrm{U}(a) \cap \mathrm{U}(b)$
- *3):* $\mathrm{U}(\Sigma a) = \mathrm{U}(a)$
- *4):* $\mathrm{U}(a) \supseteq \mathrm{U}(b) \cap \mathrm{U}(c)$ *for every distinguished triangle* $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$
- *5):* $\mathrm{U}(a \otimes b) = \mathrm{U}(a) \cup \mathrm{U}(b)$

Proof

Recall that a prime ideal is in particular: proper, additive, triangulated, thick and a $\otimes$*-ideal.*

- *1):* $0 \in \mathcal{P}$ *for every* $\mathcal{P}$ *(additive);* $1 \in \mathcal{P} \implies \mathcal{P} = \mathcal{K}$ *(*$\otimes$*-ideal) contradicting properness*
- *2):* *"*$\subseteq$*": thick and "*$\supseteq$*": additive*
- *4):* *triangulated (note: works also for any permutation of* a, b, c*)*
- *3):* *Apply 4) twice to* $a \longrightarrow 0 \longrightarrow \Sigma a \longrightarrow \Sigma a$ *and use 1)*
- *5):* *"*$\subseteq$*": prime and "*$\supseteq$*":* $\otimes$*-ideal*

Remark

Define $\mathrm{supp}(a) := Z(\{a\}) = \mathrm{Spc}(\mathcal{K}) \setminus \mathrm{U}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) |\ a \notin \mathcal{P}\}$*, then this satisfies the "dual" ("complementary") properties for closed sets.*

Definition

A support data on a tensor triangulated category $\mathcal{K}$ is a pair (X, σ) of a topological space X and a closed subset $\sigma(a) \subseteq X$ for any $a \in \mathcal{K}$ such that:

1): $\sigma(0) = \emptyset$ and $\sigma(1) = X$

2): $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$

3): $\sigma(\Sigma a) = \sigma(a)$

4): $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$ for every distinguished triangle $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$

5): $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$

A morphism $f\colon (X, \sigma) \to (Y, \tau)$ of support data on $\mathcal{K}$ is a continuous map $f\colon X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$.

Theorem

Let $\mathcal{K}$ be a tensor triangulated category. The spectrum $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is the final support data on $\mathcal{K}$, i.e. for any support data (X, σ) on $\mathcal{K}$ there exists a unique continuous map $f\colon X \to \mathrm{Spc}(\mathcal{K})$ such that $\sigma(a) = f^{-1}(\mathrm{supp}(a))$.

Remark

The map $f\colon X \to \mathrm{Spc}(\mathcal{K})$ above can be given explicitly by:

$$f(x) = \{a \in \mathcal{K} \mid x \notin \sigma(a)\} \qquad \forall x \in X.$$

Assume that the tensor triangulated category $\mathcal{K}$ is also rigid and idempotent-complete.

Definition

For an open set $U \subseteq \mathrm{Spc}(\mathcal{K})$ denote its closed complement by $Z := \mathrm{Spc}(\mathcal{K}) \setminus U$.
Define the thick $\otimes$-ideal $\mathcal{K}_Z := \{a \in \mathcal{K} | \ \mathrm{supp}(a) \subseteq Z\}$ (follows from support data properties).
Define $\mathcal{K}(U) := (\mathcal{K}/\mathcal{K}_Z)^\natural$ the idempotent completion of the Verdier quotient.

Remark

The Verdier quotient $\mathcal{K}/\mathcal{J}$ is localizing with respect to all morphisms, whose cone lies in $\mathcal{J}$.
The following holds: $(\mathcal{K}(U))(V) \cong \mathcal{K}(V)$ for every $V \subseteq U \cong \mathrm{Spc}(\mathcal{K}(U))$.
The ring $\mathrm{End}_{\mathcal{K}(U)}(\mathbb{1}_{\mathcal{K}(U)})$ is commutative (product given by the tensor product).

Definition

Define a presheaf of commutative rings on $\mathrm{Spc}(\mathcal{K})$ by

$$U \mapsto \mathrm{End}_{\mathcal{K}(U)}(\mathbb{1}_{\mathcal{K}(U)})$$

on a basis of quasi-compact open subsets.
Define the structure sheaf $\mathcal{O}_{\mathcal{K}}$ as the sheafification of this presheaf and denote by

$$\mathrm{Spec}(\mathcal{K}) := (\mathrm{Spc}(\mathcal{K}), \mathcal{O}_{\mathcal{K}})$$

the locally ringed space (not known to be a scheme).